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Adaptive synchronization of chaotic systems via linear balanced feedback control

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Abstract

Synchronization of a chaotic rigid body system via adaptive linear balanced feedback control is investigated in this paper. Firstly, we obtain feedback gains by linear feedback control scheme based on Lyapunov stability theory and constrained extreme strategy. Next, using the result of the analysis, an adaptive linear balanced feedback controller is designed for chaos synchronization. The proposed scheme can be implemented without requiring the upper bound of the trajectory of a chaotic system in advance. Numerical simulations are provided to verify the effectiveness and feasibility of the designed synchronization schemes.

1. Introduction

Chaos is a very interesting nonlinear phenomenon and has been studied extensively over the past years in many physical systems that possess nonlinearity [1,2]. Because its characteristic is extremely sensitive dependent on initial conditions, chaotic systems are difficult to be synchronized.

Since Pecora and Carroll [3] proposed the PC method for synchronizing two identical chaotic systems with different initial conditions, chaos synchronization has been widely explored in a variety of fields including secure communications, optics, chemical and biological systems and so on. In past years, various methods have been developed for the synchronization of chaotic systems such as linear feedback control [4], backstepping design [5], active control [6], nonlinear control [7], adaptive control [8–14], etc. In some control schemes, it is essential to know the upper bounds of the trajectory of a chaotic system in advance. In a practical situation, the values of these upper bounds are unknown. Hence, a modified controller for a chaos synchronization system without predetermining the upper bound of system states is an important problem.

In this paper, a new approach combining both linear balanced feedback gain and adaptive controller is proposed. Based on the linear feedback scheme and extreme approach, the roughly balanced feedback gains of the system can be obtained analytically and the convergent rate of state error dynamics is roughly balanced with respect to each state error. Next, to implement this adaptive balanced feedback gain, an adaptation law is adopted to estimate the upper bound of the trajectory of the chaotic system. This work will provide a detailed

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design process for chaos synchronization later. A rigid body chaotic system is demonstrated as an example to show the effectiveness of the proposed method.

Euler's rigid body system with three quadratic nonlinear terms is a simple and important three-dimensional autonomous system in classical mechanics. For certain linear feedback gains, Euler's rigid body system has two strange attractors [15] and two-scroll chaotic attractors [16]. In addition, Ref. [17] presents another three-dimensional autonomous chaotic system, which can display two- and four-scroll attractors. The foregoing two systems are equivalent representations of the system if state transformation exists. By state transformation, Euler's rigid body can transfer to the corresponding Liu's system [17]. Then, by the addition of appropriate feedback gains, the rigid body system can also generate a four-scroll chaotic attractor.

2. Design of controller

In this section, a systematic design process of adaptive synchronization of two identical chaotic systems is provided via linear balanced feedback scheme and adaptation law.

2.1. Nonadaptive design based on linear balanced feedback control

First, we design a controller based on linear feedback control and minimization of the sum of the feedback gains to synchronize chaotic systems.

Consider a chaotic system in the form of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in R^n$ is the state vector, $\mathbf{A} \in R^{n\times n}$ is a constant matrix, and $\mathbf{F}(\mathbf{x})$ is a continuous nonlinear function. System (1) is considered as a drive system.

From the linear feedback approach, the controlled response system is given by

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{F}(\mathbf{y}) - \mathbf{K}(\mathbf{y} - \mathbf{x}), \tag{2}$$

where $\mathbf{y} \in R^n$ denotes the state vector of the response system, and $\mathbf{K} = \mathrm{diag}\{k_1, k_2, \ldots, k_n\} \in R^{n\times n}$ is a feedback matrix to be designed later.

The dynamics of synchronization errors can be expressed as

$$\dot{\mathbf{e}} = \mathbf{B}\mathbf{e} + \mathbf{F}(\mathbf{e}), \tag{3}$$

where $\mathbf{e} = \mathbf{y} - \mathbf{x} \in R^n$ is the state error vector, $\mathbf{B} = \mathbf{A} + \mathbf{J} - \mathbf{K}$,

$$\mathbf{J} = \left.\frac{\partial \mathbf{F}(\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{x}} \in R^{n\times n},$$

is the Jacobian matrix evaluated at $\mathbf{y} = \mathbf{x}$.

Construct a Lyapunov function

$$V(\mathbf{e}) = \mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{e}, \tag{4}$$

where $\mathbf{P}$ is a positive definite diagonal constant matrix.

The derivative of the Lyapunov function along the trajectory of system (3):

$$\dot{V} = \dot{\mathbf{e}}^{\mathrm{T}}\mathbf{P}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} = \mathbf{e}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{B})\mathbf{e} + (\mathbf{F}^{\mathrm{T}}\mathbf{P}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{F}) = -\mathbf{e}^{\mathrm{T}}\mathbf{Q}\mathbf{e} \leqslant -\mathbf{E}^{\mathrm{T}}\mathbf{M}\mathbf{E}, \tag{5}$$

where $\mathbf{Q} \in R^{n\times n}$ is a positive definite matrix of variables $\mathbf{x}$, and $\mathbf{M} \in R^{n\times n}$ is a positive definite constant matrix. Assume that there exists a positive definite matrix $\mathbf{P}$ such as $\mathbf{F}^{\mathrm{T}}\mathbf{P}\mathbf{e} = \mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{F} = 0$ in this system. In fact, most of chaotic systems, including Lorenz, Lü, Chen and four-scroll new chaotic systems can be described by this expression.

Chaos synchronization problem is to design linear feedback gain matrix $\mathbf{K}$ to make the matrix $\mathbf{M}$ a positive definite function. Then the states of the response system and drive system are globally asymptotically synchronized. To implement balanced feedback gains, a method minimizing the sum of the feedback gains is adopted to obtain a set of roughly equal control gains. The procedure for designing control gains is described as follows:

The first step is to solve the linear feedback control gains from the positive definite matrix $\mathbf{M}$. Assume all the principal minor determinants corresponding to the symmetric matrix $\mathbf{M}$ as the following:

$$\Delta_i = \left|M_{qr}\right| = m_i > 0, \quad q, r = 1, 2, \ldots, i, \quad i = 1, 2, \ldots, n. \tag{6}$$

From Eq. (6), we obtain

$$k_i = S_i(m_1, m_2, \ldots, m_q), \quad q = 1, 2, \ldots, i, \quad i = 1, 2, \ldots, n \tag{7}$$

and

$$\frac{\partial k_i}{\partial m_i} \neq 0, \quad \frac{\partial k_j}{\partial m_i} = 0, \quad j = 1, 2, \ldots i-1, \quad i = 1, 2, \ldots, n. \tag{8}$$

The second step is to minimize the sum of the control gains, i.e., $f = \mathrm{Min}(k_1 + \cdots + k_n)$. This means that the control gains are roughly equal, i.e. balanced.

The third step is to study the minima of function of specific variables. Then, write down the necessary conditions for rendering f a relative maximum or minimum as follows:

$$\frac{\partial f}{\partial m_i} = 0, \quad i = 1, 2 \ldots, n. \tag{9}$$

By solving Eq. (9) corresponding to Eq. (8), the extreme point $(m_1^*, m_2^*, \ldots, m_n^*)$ is found.

2.2. Modification of a controller based on adaptation law

The controller designed in Section 2.1 requires the knowledge of the system states to estimate the value of upper bound of the chaotic system. However, in many practical situations, it is difficult to exactly determine the values of the trajectory in advance. Therefore, adaptive synchronization of two chaotic systems is essential. In this section, we consider the problem of adaptive synchronization of two identical chaotic systems with uncertain feedback gains. By using linear balanced feedback control, a relation about feedback gain matrix is obtained. Then, an adaptive feedback gain replaces this balanced feedback gain matrix. So, from Eq. (2), the controlled response system can be reconstructed as

$$\dot{\hat{\mathbf{y}}} = \mathbf{A}\hat{\mathbf{y}} + \mathbf{F}(\hat{\mathbf{y}}) - \hat{\mathbf{K}}(\hat{\mathbf{y}} - \mathbf{x}), \tag{10}$$

where $\hat{\mathbf{y}} \in R^n$ denotes the estimated state vector of the response system, and $\hat{\mathbf{K}} = \mathrm{diag}\{\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_n\} \in R^{n\times n}$ is an estimated balanced feedback matrix.

The dynamics of adaptive synchronization errors can be expressed as

$$\dot{\mathbf{s}} = \mathbf{B}\mathbf{s} + \mathbf{F}(\mathbf{s}) - \tilde{\mathbf{K}}\,\mathbf{s}, \tag{11}$$

where $\mathbf{s} = \hat{\mathbf{y}} - \mathbf{x} = [s_1, s_2, \ldots, s_n]^{\mathrm{T}}$, $\mathbf{B} = \mathbf{A} + \mathbf{J} - \mathbf{K}$,

$$\mathbf{J} = \left.\frac{\partial \mathbf{F}(\hat{\mathbf{y}})}{\partial \hat{\mathbf{y}}}\right|_{\hat{\mathbf{y}}=\mathbf{x}},$$

is the Jacobian matrix evaluated at $\hat{\mathbf{y}} = \mathbf{x}$, and $\tilde{\mathbf{K}} = \hat{\mathbf{K}} - \mathbf{K}$.

Choose the Lyapunov function candidate

$$V_1(\mathbf{s}, \hat{\mathbf{K}}) = \mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s} + (\tilde{\mathbf{K}}\,\mathbf{I}_{n\times 1})^{\mathrm{T}}\mathbf{R}(\tilde{\mathbf{K}}\,\mathbf{I}_{n\times 1}), \tag{12}$$

where $\mathbf{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_n\}$, $\mathbf{R} = \mathrm{diag}\{r_1, r_2, \ldots, r_n\}$ are positive definite diagonal constant matrices, and $\mathbf{I}_{n\times 1} = [1,1,\ldots,1]^{\mathrm{T}}$ is an unitary vector. Then, its derivative is

$$\begin{aligned}\dot{V}_1(\mathbf{s}, \hat{\mathbf{K}}) &= \dot{\mathbf{s}}^{\mathrm{T}}\mathbf{P}\mathbf{s} + \mathbf{s}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{s}} + (\dot{\tilde{\mathbf{K}}}\,\mathbf{I}_{n\times1})^{\mathrm{T}}\mathbf{R}(\tilde{\mathbf{K}}\,\mathbf{I}_{n\times1}) + (\tilde{\mathbf{K}}\,\mathbf{I}_{n\times1})^{\mathrm{T}}\mathbf{R}(\dot{\tilde{\mathbf{K}}}\,\mathbf{I}_{n\times1})\\ &= \mathbf{s}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{B})\mathbf{s} + (\mathbf{F}^{\mathrm{T}}\mathbf{P}\mathbf{s} + \mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{F}) - \mathbf{s}^{\mathrm{T}}(\tilde{\mathbf{K}}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\tilde{\mathbf{K}})\mathbf{s} + 2(\tilde{\mathbf{K}}\,\mathbf{I}_{n\times1})^{\mathrm{T}}\mathbf{R}(\dot{\tilde{\mathbf{K}}}\,\mathbf{I}_{n\times1})\\ &= -\mathbf{s}^{\mathrm{T}}\mathbf{Q}\mathbf{s} - 2\mathbf{s}^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{K}}\,\mathbf{s} + 2(\tilde{\mathbf{K}}\,\mathbf{I}_{n\times1})^{\mathrm{T}}\mathbf{R}(\dot{\tilde{\mathbf{K}}}\,\mathbf{I}_{n\times1}),\end{aligned} \tag{13}$$

where $\mathbf{F}^{\mathrm{T}}\mathbf{P}\mathbf{s} = \mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{F} = 0$.

Select

$$-2\mathbf{s}^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{K}}\,\mathbf{s} + 2(\tilde{\mathbf{K}}\,\mathbf{I}_{n\times1})^{\mathrm{T}}\mathbf{R}(\dot{\tilde{\mathbf{K}}}\,\mathbf{I}_{n\times1}) = 0, \tag{14}$$

where the estimated value of linear balanced feedback gain matrix is in the form of

$$\hat{\mathbf{K}} = \mathrm{diag}\{\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_n\}, \quad \hat{k}_i = a_{0i} + a_{1i}\hat{X}, \quad i = 1, \ldots, n, \tag{15}$$

the coefficients a_{0i} and a_{1i} can be specified by linear balanced feedback approach. $\hat{X}$ is the estimated maximum upper bound of the absolute values of system states.

Then,

$$\dot{V}_1(\mathbf{s}, \hat{X}) = -\mathbf{s}^{\mathrm{T}}\mathbf{Q}\mathbf{s} \leqslant -\mathbf{S}^{\mathrm{T}}\mathbf{M}\mathbf{S}, \tag{16}$$

where $\mathbf{S} = [|s_1|, |s_2|, \ldots |s_n|]^{\mathrm{T}}$ and $\mathbf{M}$ is a positive definite constant function, which is negative semi-definite function of the state error $\mathbf{s}$ and $\hat{X}$. By partial stability theory [13], the partial variables $\mathbf{s}$ in Eq. (11) are asymptotically stable about $\mathbf{s} = 0$, the synchronization manifold is stable. Hence the drive and response systems can be synchronized.

From Eq. (14), we can obtain the adaptation law of the form

$$\dot{\hat{X}} = \sum_{i=1}^{n} p_i a_{1i} s_i^2 \Big/ \sum_{i=1}^{n} r_i a_{1i}^2. \tag{17}$$

The convergence properties of adaptive control system (11) can be described as the state error $\mathbf{s}$ converges to the origin with a rate of at least $\gamma/2$ proofed in the Appendix where $\gamma = [\lambda_{\min}(\mathbf{M}) - \lambda_{\max}(-\mathbf{P}\tilde{\mathbf{K}})]/\lambda_{\max}(\mathbf{P})$. Finally, the convergent rate of the steady-state error $\mathbf{s}$ is at least $\gamma_f/2$ where $\gamma_f = \gamma|_{t=\infty} = \lambda_{\min}(\mathbf{M})/\lambda_{\max}(\mathbf{P})$. In addition, the convergence properties of corresponding adaptation law can be described as that $\dot{\hat{X}}$ converges to the origin with a rate of at least γ where $\gamma_0 \leqslant \gamma \leqslant \gamma_f$. When $\gamma|_{t=\infty} = \gamma_f$, the convergent rate of the steady-state adaptation $\dot{\hat{X}}$ is the largest. When $\gamma|_{t=0} = \gamma_0$, the convergent rate of the initial adaptation $\dot{\hat{X}}$ is the smallest.

3. Adaptive synchronization of two identical chaotic systems

In this section, we take a rigid body system to the Liu's chaotic system [17] to create a four-scroll chaotic attractor. In addition, adaptive linear balanced feedback control is used to achieve synchronization of two identical rigid body chaotic systems.

3.1. Equation of motion

Euler's equations for a rigid body motion with linear feedback control are:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + G_1,\\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + G_2,\\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + G_3,\end{aligned} \tag{18}$$

where I_1, I_2, I_3 are the principle moments of inertia with respect to body axes, $\omega_1, \omega_2, \omega_3$ are the angular velocities about principle axes fixed at the center of mass and G_1, G_2, G_3 are the three control torque. Without loss of generality, we assume $I_3 > I_1 > I_2$.

Let torque feedback matrix $\mathbf{G} = \mathbf{H}\boldsymbol{\omega}$, where

$$\mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 0 \\ 0 & 0 & h_{33} \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{19}$$

By a change of coordinates $\mathbf{x} = \mathrm{T}(\boldsymbol{\omega})$, the state-space model in $\mathbf{x} = [x_1, y_1, z_1]^{\mathrm{T}}$ is as follows:

$$\begin{aligned} \dot{x}_1 &= ax_1 + d_1 y_1 z_1, \\ \dot{y}_1 &= by_1 + d_2 x_1 z_1, \\ \dot{z}_1 &= cz_1 + d_3 x_1 y_1, \end{aligned} \tag{20}$$

where $d_1 < 0$, $d_2 > 0$, $d_3 > 0$ and

$$\mathbf{x} = \mathrm{T}(\boldsymbol{\omega}) = \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \end{bmatrix} = \begin{bmatrix} (I_3 - I_1)(I_1 - I_2)/(d_2 d_3 I_2 I_3) \\ (I_1 - I_2)(I_2 - I_3)/(d_1 d_3 I_1 I_3) \\ (I_2 - I_3)(I_3 - I_1)/(d_1 d_2 I_1 I_2) \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} I_1 a & 0 & 0 \\ 0 & I_2 b & 0 \\ 0 & 0 & I_3 c \end{bmatrix}. \tag{21}$$

After this state transformation, a rigid body motion with appropriate feedback gains can generate four-scroll chaotic attractors. For example, $I_1 = 2I_0$, $I_2 = I_0$, $I_3 = 3I_0$, $d_1 = -1$, $d_2 = 1$, $d_3 = 1$, and $a = 0.5$, $b = -10$, $c = -4$, we obtain $x_1 = \sqrt{3}\omega_1/3$, $x_2 = \sqrt{3}\omega_2/3$, $x_3 = \omega_3$, and $h_{11} = I_0$, $h_{22} = -10I_0$, $h_{33} = -12I_0$. Now, for system (18), one can observe in simulation a four-scroll chaotic attractor as shown in Fig. 1.

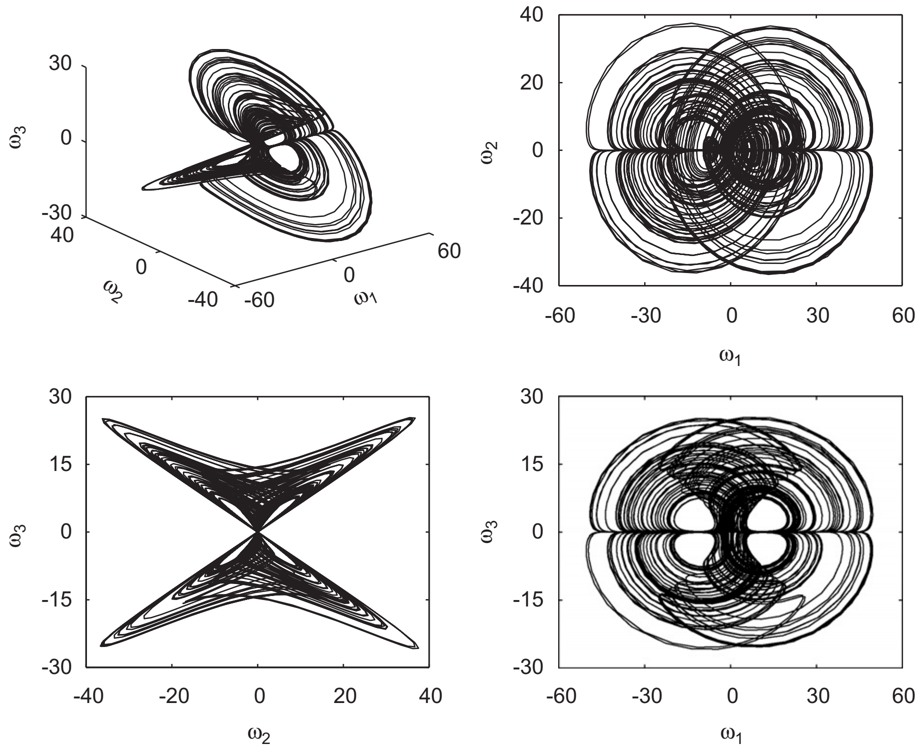

Fig. 1. The four-scroll chaotic attractor of a rigid body motion with linear feedback control in $\boldsymbol{\omega}$-space at $h_{11} = I_0$, $h_{22} = -10I_0$, $h_{33} = -12I_0$.

3.2. Adaptive control

Let us consider four-scroll chaotic system (20) as the drive system and its controlled response system is described as

$$\begin{aligned}\dot{x}_2 &= ax_2 + d_1y_2z_2 - k_1(x_2 - x_1),\\ \dot{y}_2 &= by_2 + d_2x_2z_2 - k_2(y_2 - y_1),\\ \dot{z}_2 &= cz_2 + d_3x_2y_2 - k_3(z_2 - z_1).\end{aligned} \tag{22}$$

Subtracting Eq. (20) from Eq. (22), we can obtain the error dynamics in the form of

$$\dot{\mathbf{e}} = \mathbf{Be} + \mathbf{F(e)}, \tag{23}$$

where

$$\mathbf{B} = \begin{bmatrix} a-k_1 & d_1z_1 & d_1y_1 \\ d_2z_1 & b-k_2 & d_2x_1 \\ d_3y_1 & d_3x_1 & c-k_3 \end{bmatrix}, \quad \mathbf{F(e)} = \begin{bmatrix} d_1e_2e_3 \\ d_2e_1e_3 \\ d_3e_1e_2 \end{bmatrix}.$$

Construct a Lyapunov function

$$V = \mathbf{e}^{\mathrm{T}}\mathbf{Pe}, \tag{24}$$

where $\mathbf{P} = \mathrm{diag}\{p_1, p_2, p_3\}$ is a positive definite matrix.

The derivative of the Lyapunov function along the trajectory of system (23):

$$\dot{V} = \dot{\mathbf{e}}^{\mathrm{T}}\mathbf{Pe} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} = \mathbf{e}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{PB})\mathbf{e} + (\mathbf{F}^{\mathrm{T}}\mathbf{Pe} + \mathbf{e}^{\mathrm{T}}\mathbf{PF}) = -\mathbf{e}^{\mathrm{T}}\mathbf{Q} \leqslant -\mathbf{E}^{\mathrm{T}}\mathbf{ME}, \tag{25}$$

where $\mathbf{F}^{\mathrm{T}}\mathbf{Pe} = \mathbf{e}^{\mathrm{T}}\mathbf{PF} = 0$, $\mathbf{E} = [|e_1||e_2||e_3|]^{\mathrm{T}}$, U_1, U_2, U_3 are the upper bounds of the absolute values of $p_1d_1x_1$, $p_2d_2y_1$, $p_3d_3z_1$ respectively, and

$$\mathbf{Q} = \begin{bmatrix} 2(k_1-a)p_1 & p_3d_3z_1 & p_2d_2y_1 \\ p_3d_3z_1 & 2(k_2-b)p_2 & p_1d_1x_1 \\ p_2d_2y_1 & p_1d_1x_1 & 2(k_3-c)p_3 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 2(k_1-a)p_1 & -U_3 & -U_2 \\ -U_3 & 2(k_2-b)p_2 & -U_1 \\ -U_2 & -U_1 & 2(k_3-c)p_3 \end{bmatrix}.$$

Obviously, to ensure that the origin of error system (23) is asymptotically stable, the matrix $\mathbf{M}$ should be positive definite. By Sylvester's theorem, all principal minors of $\mathbf{M}$ are strictly positive, i.e., a suitable linear feedback gain matrix $\mathbf{K}$ can be chosen if the following conditions hold:

$$\begin{aligned} k_1 &= a + m_1/(2p_1),\\ k_2 &= b + (m_2 + U_3^2)/(2m_1p_2),\\ k_3 &= c + [m_1m_3 + (m_1U_1 + U_3U_2)^2 + m_2U_2^2]/(2m_1m_2p_3). \end{aligned} \tag{26}$$

where m_1, m_2, m_3 are positive constants, $U_1 = |p_1d_1|U_x$, $U_2 = |p_2d_2|U_y$, $U_3 = |p_3d_3|U_z$; U_x, U_y and U_z are the upper bounds of the absolute values of variables x_1, y_1 and z_1, respectively.

Remark. Define X is the maximum upper bound of the absolute values of variables x_1, y_1 and z_1, then $U_1 = |p_1d_1|X$, $U_2 = |p_2d_2|X$, $U_3 = |p_3d_3X$, where X can be specified by solving Eq. (20).

To obtain balanced feedback gains, a method minimizing the sum of the feedback gains is presented as follows. The minimization of the functional $f = \mathrm{Min}\{k_1 + k_2 + k_3\}$ is required. From Eq. (8) $\partial k_3/\partial m_3 \neq 0$, $\partial k_j/\partial m_3 = 0$, $j = 1,2$, we know that the necessary condition $\partial f/\partial m_3 = 0$ fails to exist. Thus, it may happen that an extreme value is taken on at a boundary point, i.e., $m_3 = 2\varepsilon p_3m_2$, $\varepsilon \to 0^+$.

In a case study, a set of parameters of the four-scroll chaotic system is defined by $d_1 = -1$, $d_2 = d_3 = 1$, $a>0$, $b<0$, $c<0$, and assuming that $p_1 = p_0$, $p_2 = p_3 = p_0/2$. Thus, we obtain $U_1 = p_0X$, $U_2 = 0.5p_0X$, $U_3 = 0.5p_0X$.

By solving the minimization of the functional $f = \mathrm{Min}\{k_1 + k_2 + k_3\}$, we can obtain linear balanced feedback gains. If $f_{m_1} = 0,\ f_{m_2} = 0$, and $f_{m_1m_1} > 0,\ f_{m_1m_1}f_{m_2m_2} > f^2_{m_1m_2}$ at a point $p(m_1^*, m_2^*)$, then at that point f has a relative minimum. With

$$f(k_1, k_2, k_3) = k_1 + k_2 + k_3 = f(m_1, m_2, 0^+), \tag{27}$$

the necessary conditions $f_{m_1} = f_{m_2} = 0$ at (m_1^*, m_2^*) become

$$f_{m_1} = -(-8m_1^2m_2 + 8p_0^2X^2m_2 + 16m_2^2 - 16p_0^2X^2m_1^2 + p_0^4X^4)/(16p_0m_1^2m_2) = 0,$$

$$f_{m_2} = -(-16m_2^2 + 16p_0^2X^2m_1^2 + 8p_0^3X^3m_1 + p_0^4X^4)/(16p_0m_1m_2^2) = 0, \tag{28}$$

from which there follows:

$$m_1^* = \sqrt{2}p_0X > 0, \quad m_2^* = (4\sqrt{2} + 1)p_0^2X^2/4 > 0 \tag{29}$$

and

$$f_{m_1m_1} \approx 1.56/(p_0^2X) > 0, \quad f_{m_1m_1}f_{m_2m_2} - f^2_{m_1m_2} \approx 0.6/(p_0^6X^4) > 0. \tag{30}$$

Thus, the corresponding minimum sum of control gains is

$$\mathrm{Min}\{k_1 + k_2 + k_3\} = (a + b + c') + (\sqrt{2} + 2)X, \quad \text{i.e.,} \tag{31}$$

$$k_1 = a + \sqrt{2}X/2, \quad k_2 = b + (\sqrt{2}/4 + 1)X, \quad k_3 = c' + (\sqrt{2}/4 + 1)X,$$

where $c' = c + \varepsilon,\ \varepsilon \to 0^+$.

To implement the adaptive controller, the adaptation law (17) is adopted and is rewritten as

$$\dot{\hat{X}} = (p_1a_{11}s_1^2 + p_2a_{12}s_2^2 + p_3a_{13}s_3^2)/(r_1a_{11}^2 + r_2a_{12}^2 + r_3a_{13}^2). \tag{32}$$

where $p_1 = p_0,\ p_2 = p_3 = p_0/2,\ a_{11} = \sqrt{2}/2,\ a_{12} = a_{13} = \sqrt{2}/4 + 1$, and assuming that $r_1 = \sigma p_1,\ r_2 = \sigma p_2,\ r_3 = \sigma p_3$.

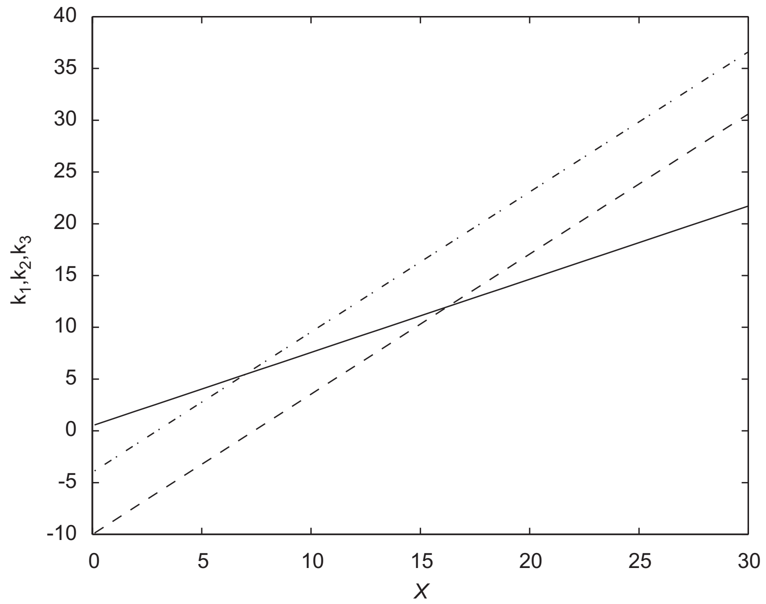

Fig. 2. Feedback gains k_1(–), k_2(−−), k_3(–.) versus X from Eq. (31).

4. Numerical results

The numerical simulations are carried out as shown in Figs. 1–6. A rigid body motion with certain feedback gains displays a four-scroll chaotic attractor in **ω**-space, as depicted in Fig. 1. In Fig. 2, it shows the analytical results of the feedback gains k_1, k_2 and k_3 versus X from Eq. (31) by the Lyapunov stability theory and extreme scheme. The initial states of the drive system (20), response system (22) and adaptive equation (32) are $x_1(0) = 1$, $y_1(0) = 1$, $z_1(0) = 1$, $x_2(0) = -10$, $y_2(0) = -17$, $z_2(0) = 15$, and $\hat{X}(0) = 0$, respectively. The states of four-scroll chaotic attractor are bounded and satisfy the inequalities: $-29.7 < x < 28.5$; $-21 < y < 22.1$ and $-26.4 < z < 22.2$. Choosing the maximum upper bound as $X = 30$ and the associated feedback gains as $K = (k_1,k_2,k_3) = (21,.71,30.61,36.61)$, one can achieve chaos synchronization at $t = 1$ with state errors $(e_1,e_2,e_3) = -(9,0.015,0.015) \times 10^{-9}$ as shown in Fig. 3.

Fig. 4 displays the time response of states x_1, y_1, z_1 for the drive system and the states x_2, y_2, z_2 for the response system with adaptive control. The dynamics of adaptive synchronization errors for the drive system and response system is shown in Fig. 5. The control is activated at $t = 3$ and the maximum adaptive synchronization error is $|s_i|_{\max} = 0.68 \times 10^{-10}$. As shown in Fig. 6, the corresponding estimated linear balanced feedback gains and maximum upper bound are $(\hat{k}_1, \hat{k}_2, \hat{k}_3) = (9.4703, 7.1711, 13.1711)$, and $\hat{X} = 12.6860$. The above results show that adaptive synchronization is achieved successfully.

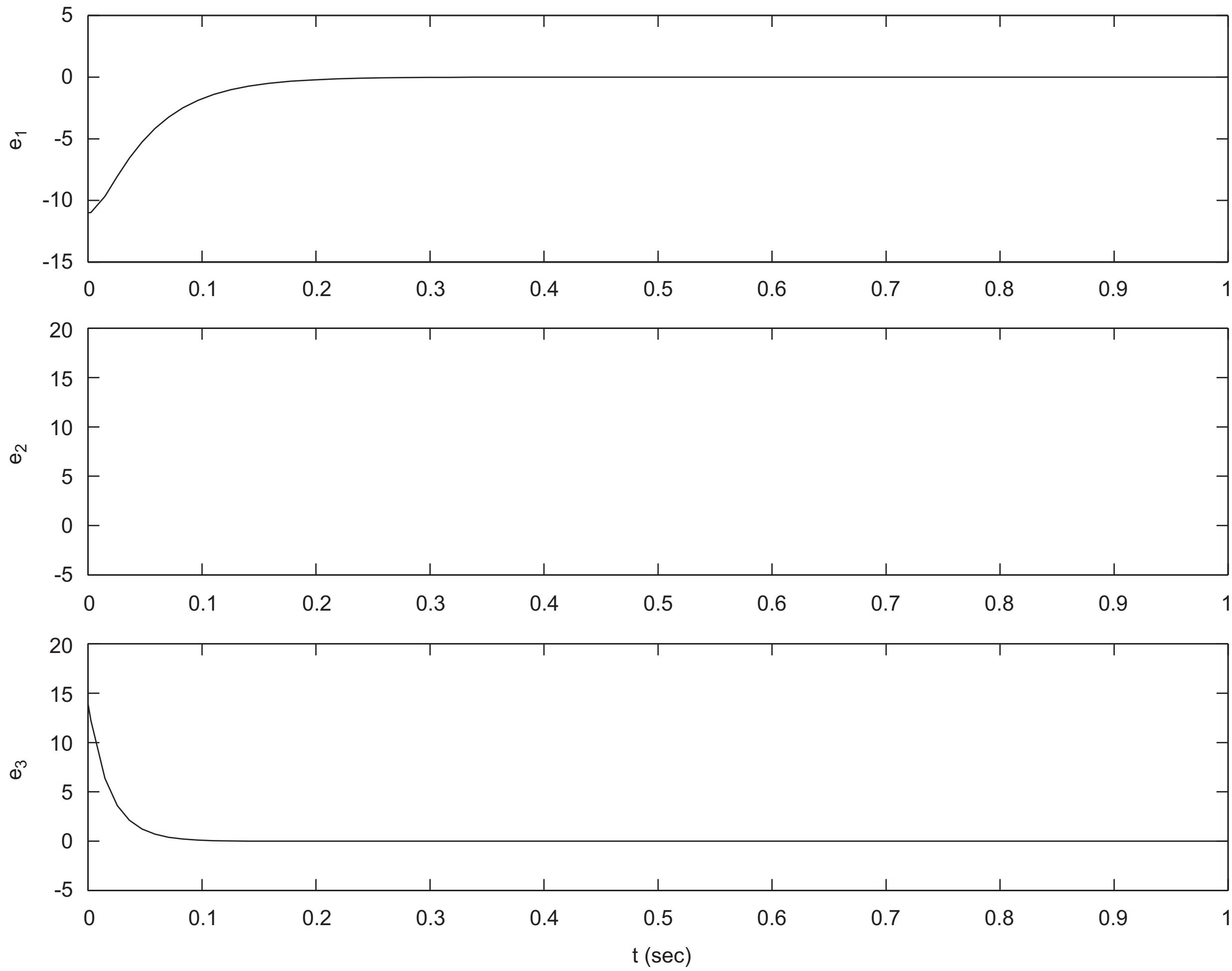

Fig. 3. Synchronization errors of two identical four-scroll chaotic systems with feedback gains (21.71, 30.61, 36.61) from Eq. (31) at $X = 30$.

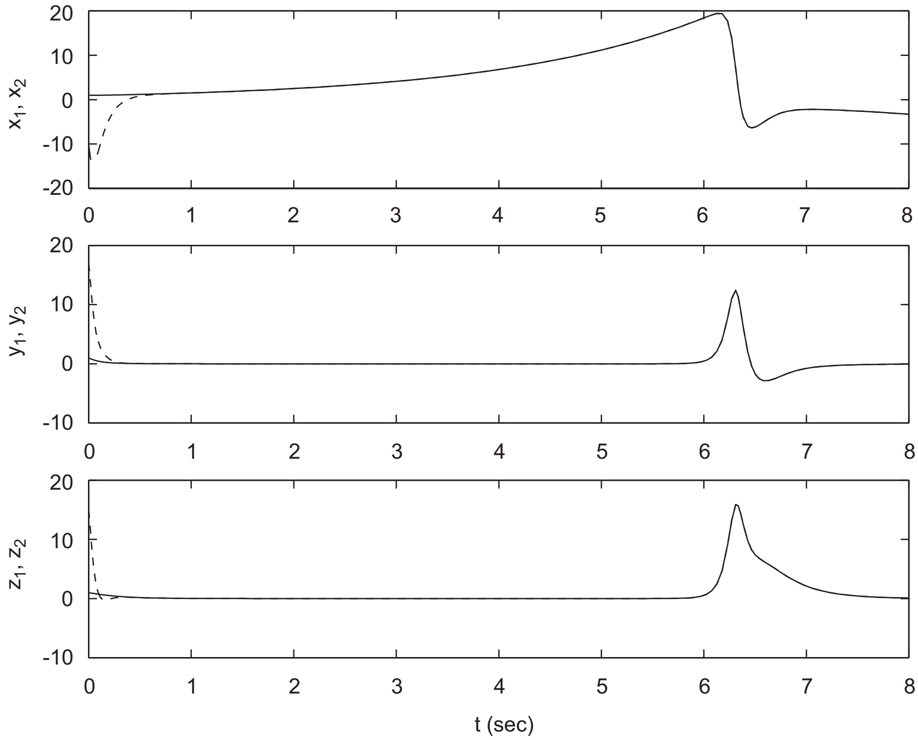

Fig. 4. The time response of the states for the drive system (20) and for the response system (22).

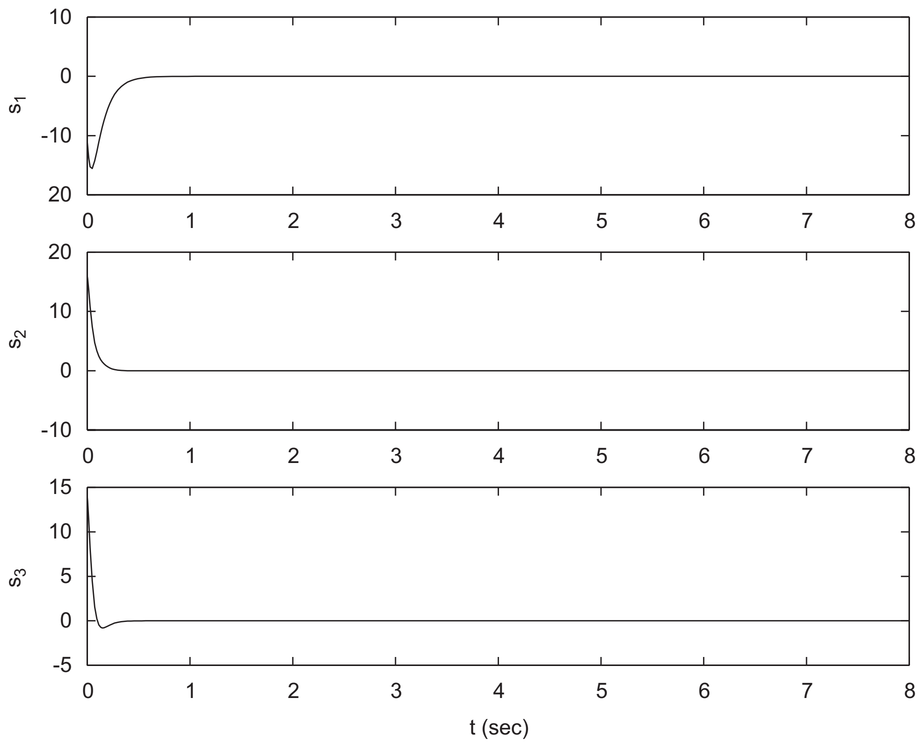

Fig. 5. Dynamics of adaptive synchronization errors (s_1,s_2,s_3) for the drive system (20) and for the response system (22).

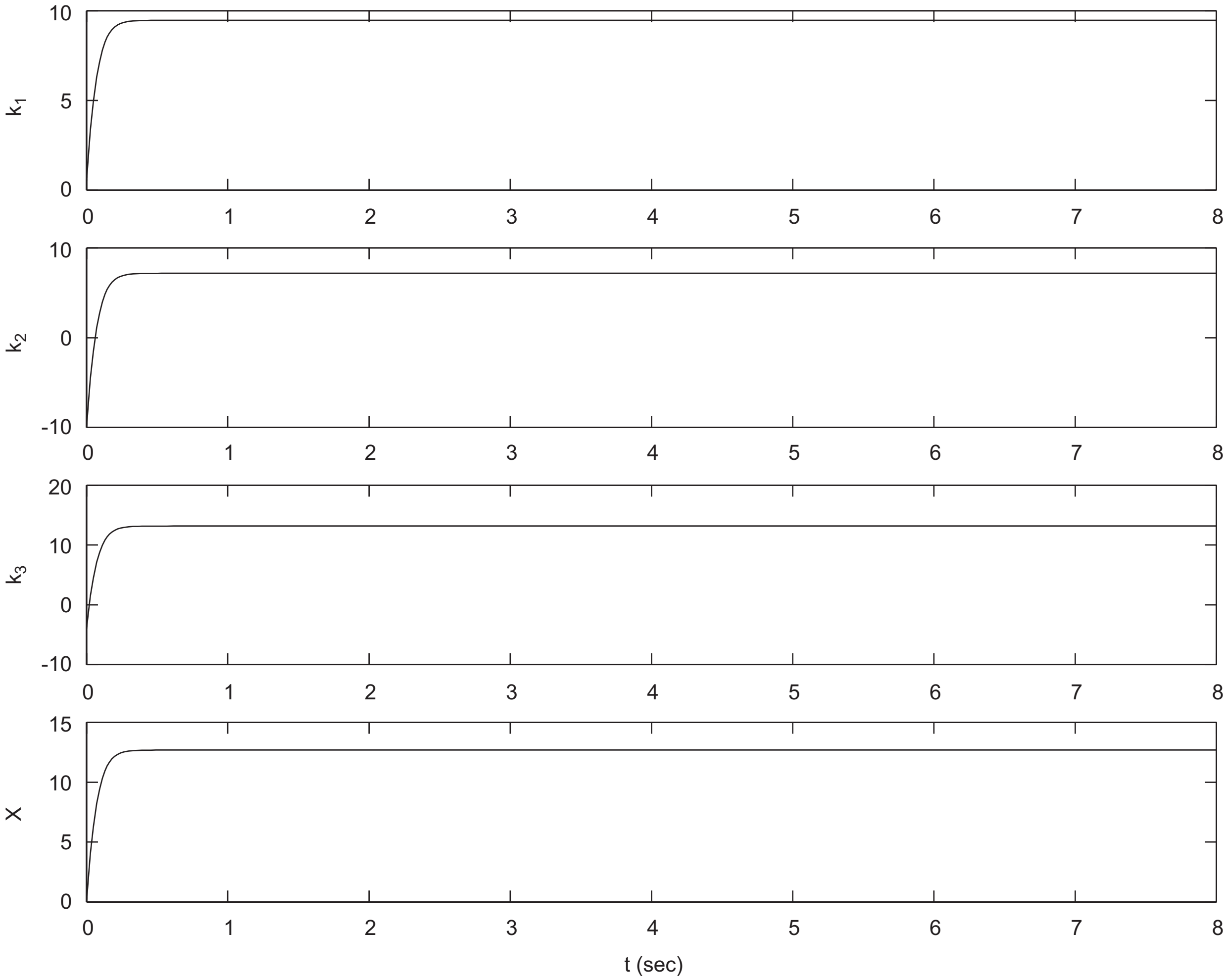

Fig. 6. Estimated linear balanced feedback gains ($\hat{k}_1, \hat{k}_2, \hat{k}_3$)and estimated upper bound $\hat{X}$ versus time.

5. Conclusion

This paper demonstrates that a rigid body motion with linear feedback control can generate a four-scroll chaotic attractor and presents a method to design an adaptive linear balanced feedback controller for chaos synchronization of two four-scroll chaotic systems without predetermining the upper bound of system states. Based on the Lyapunov stability theory, an adaptive controller is designed for estimating balanced feedback gains and the maximum upper bound of system states. Furthermore, the convergent rates of the steady-state error $\mathbf{s}$ and the steady-state adaptation $\dot{\hat{X}}$ are derived at least $\gamma_f/2$ and γ_f, respectively. Finally, numerical experiment shows the effectiveness of the proposed method.

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Appendix

The following fact and lemma are needed to derive the main result.

Fact. *For symmetric matrix* $\mathbf{P}$ *and any vector* $\mathbf{s}$, *the following inequality holds*:

$$\lambda_{\min}(\mathbf{P})\mathbf{s}^{\mathrm{T}}\mathbf{s} \leqslant \mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s} \leqslant \lambda_{\max}(\mathbf{P})\mathbf{s}^{\mathrm{T}}\mathbf{s}, \tag{A.1}$$

where $\lambda_{\min}(\cdot)$ *denotes the smallest eigenvalue of given matrix and* $\lambda_{\max}(\cdot)$ *the largest.*

Lemma. *If a real function* $W(t)$ *satisfies the inequality*

$$\dot{W}(t) + \alpha W(t) \leqslant 0, \tag{A.2}$$

where α *is a real number. Then*,

$$W(t) \leqslant W(0)\,\mathrm{e}^{-\alpha t}. \tag{A.3}$$

Now let us evaluate the convergence rate of adaptive control system (11) based on the Lyapunov analysis. From Eqs. (13) and (14), we have

$$\dot{V}_1(\mathbf{s}, \hat{\mathbf{K}}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s}) + 2(\tilde{\mathbf{K}}\,\mathbf{I}_{n\times 1})^{\mathrm{T}}\mathbf{R}(\dot{\hat{\mathbf{K}}}\,\mathbf{I}_{n\times 1}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s}) + 2\mathbf{s}^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{K}}\,\mathbf{s} = -\mathbf{s}^{\mathrm{T}}\mathbf{Q}\mathbf{s} \leqslant -\mathbf{s}^{\mathrm{T}}\mathbf{M}\mathbf{s}. \tag{A.4}$$

Thus,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s}) &\leqslant -\mathbf{s}^{\mathrm{T}}\mathbf{M}\mathbf{s} - 2\mathbf{s}^{\mathrm{T}}\mathbf{P}\tilde{\mathbf{K}}\,\mathbf{s} \leqslant -\lambda_{\min}(\mathbf{M})\mathbf{s}^{\mathrm{T}}\mathbf{s} + \lambda_{\max}(-\mathbf{P}\tilde{\mathbf{K}})\mathbf{s}^{\mathrm{T}}\mathbf{s} \\ &= -[\lambda_{\min}(\mathbf{M}) - \lambda_{\max}(-\mathbf{P}\tilde{\mathbf{K}})]\{\mathbf{s}^{\mathrm{T}}[\lambda_{\max}(\mathbf{P})\mathbf{I}\mathbf{s}]\}/\lambda_{\max}(\mathbf{P}) \leqslant -\gamma(\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s}), \end{aligned} \tag{A.5}$$

where $\gamma = [\lambda_{\min}(\mathbf{M}(X)) - \lambda_{\max}(-\mathbf{P}\tilde{\mathbf{K}})]/\lambda_{\max}(\mathbf{P})$. According to the lemma, Eq. (A.5) means that $\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s} \leqslant V_2(0)\,\mathrm{e}^{-\gamma t}$ where $V_2(0) = (\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s})_{t=0}$. This, together with the fact $\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s} \geqslant \lambda_{\min}(\mathbf{P})||\mathbf{s}(t)||^2$, implies that the state error $\mathbf{s}$ converges to the origin with a rate of at least $\gamma/2$. Finally, the convergent rate of the steady-state error $\mathbf{s}$ is at least $\gamma_f/2$ where $\gamma_f = \gamma|_{t=\infty} = \lambda_{\min}(\mathbf{M}(X))/\lambda_{\max}(\mathbf{P})$.

Since the adaptation law is

$$\dot{\hat{X}} = \sum_{i=1}^{n} p_i a_{1i} s_i^2 \Bigg/ \sum_{i=1}^{n} r_i a_{1i}^2 = \sum_{i=1}^{n} p_i a_{1i} s_i^2 \Bigg/ \left(\sigma \sum_{i=1}^{n} p_i a_{1i}^2\right), \tag{A.6}$$

where $r_i = \sigma p_i$, $\hat{X}$ is an increasing function of t. Therefore, the final estimated upper bound approaches to the upper bound of system states i.e., $\hat{X}_f = \hat{X}_{t=\infty} = X \geqslant \hat{X}(t)$.

From Eq. (15), we have $\tilde{\mathbf{K}} = \hat{\mathbf{K}} - \mathbf{K} = \mathrm{diag}\{\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\}$, $\tilde{k}_i = a_{1i}(\hat{X} - X)$ i.e., that $\tilde{\mathbf{K}}$ is a negative definite matrix. Then,

$$\lambda_{\max}(-\mathbf{P}\tilde{\mathbf{K}}) = \lambda_{\max}(-\mathrm{diag}\{p_i a_{1i}(\hat{X} - X)\}) = (p_i a_{1i})_{\max}(X - \hat{X}), \tag{A.7}$$

$$\gamma = [\lambda_{\min}(\mathbf{M}) - \lambda_{\max}(-\mathbf{P}\tilde{\mathbf{K}})]/\lambda_{\max}(\mathbf{P}) = \alpha_1 - \alpha_2(X - \hat{X}), \tag{A.8}$$

$$\gamma_0 = \gamma|_{t=0} = \alpha_1 - \alpha_2(X_f - \hat{X}_0), \tag{A.9}$$

where $\alpha_1 = \gamma_f = \lambda_{\min}(\mathbf{M})/\lambda_{\max}(\mathbf{P})$, $\alpha_2 = (p_i a_{1i})_{\max}/\lambda_{\max}(\mathbf{P})$.

Let $\beta = (a_{1i})_{\max}/\sum_{i=1}^{n} p_i a_{1i}^2$. Then, the adaptation law Eq. (A.6) is

$$\dot{\hat{X}} \leqslant (\beta/\sigma)\sum_{i=1}^{n} p_i s_i^2 = (\beta/\sigma)\mathbf{s}^{\mathrm{T}}\mathbf{P}\mathbf{s} \leqslant (\beta/\sigma)V_2(0)\,\exp(-\gamma t). \tag{A.10}$$

The convergence properties of corresponding adaptation law can be described as that $\dot{\hat{X}}$ converges to the origin with a rate of at least γ where $\gamma_0 \leqslant \gamma \leqslant \gamma_f$. When $\gamma|_{t=\infty} = \gamma_f$, the convergent rate of the steady-state adaptation $\dot{\hat{X}}$ is the largest. When $\gamma|_{t=0} = \gamma_0$, the convergent rate of the initial adaptation $\dot{\hat{X}}$ is the smallest.

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